Int. J. Solids Structures, 1978, Vol. 14, pp. 15–38. Pergamon Press. Printed in Great Britain

FLOW OF SOLIDS DURING FORMING AND EXTRUSION: SOME ASPECTS OF NUMERICAL SOLUTIONS

O. C. Zienkiewicz, P. C. Jain† and E. Oñate
Department of Civil Engineering, University of Wales, Swansea, Wales

(*Received* 8 *December* 1976; *revised* 13 *July* 1977)

Abstract—In the extrusion and forming of solids the plastic (or viscoplastic) deformations are so large that the elastic strain is negligible. The problem thus becomes one of incompressible viscous, non-Newtonian flow with prescribed boundary velocities. Various formulations of such problems (analogous to those of incompressible solid mechanics) are possible, and the paper investigates two basic processes.

Details of application to some examples of steady state flow in extrusion, drawing and rolling are given and transient free surface solutions are demonstrated for stretch forming and deep drawing. The formulation is shown to be capable of dealing with boundary friction and strain hardening. The coupling with thermal effects is demonstrated in the last section of the paper, and in addition, some practical problems of elastic spring-back which occur on the removal of load are discussed.

1. INTRODUCTION

When is a solid not a solid? The answer to that question could well be "when it is incapable of supporting deviatoric stresses without motion". Clearly if we neglect the elastic strains and consider creep strains given by a constitutive law which defines the strain rate as a non zero function of stresses

$$\dot{\epsilon}_{ij} = f(\sigma_{ij}) \tag{1}$$

then this condition is satisfied.

In plastic or viscoplastic deformation of a solid frequently the total strains are so large that the above condition is approximately satisfied, and in such cases the solid can be treated simply as a *viscous fluid of a non-Newtonian kind*. Practical examples of such flow are extremely important in forming processes of various kinds applied to metals, plastics or glasses and numerical-finite element solutions are of great significance as alternatives of general applicability are just not available. The classical slip line approach[1] has to date represented the main line of attack but its limitations to simple two dimensional forms with constant yields conditions is serious. The flow approach on the other hand presents an alternative which is computationally easy to implement and which is capable of dealing with almost any practical situation. A formulation of this type was first presented in the work of Goon *et al.*[2] and a more general numerical solution given by Zienkiewicz and Godbole for viscoplastic materials[3–7] Lee and Kobayeshi[8] for pure plasticity and simultaneously by Cornfield and Johnson[9] for an exponential "creep" type law. Recent work by Alexander and Price[10] and Dunham[11] shows some extensions of these applications.

The object of the present paper is thus three fold. First to compare some alternative formulations with regard to their accuracy, second to survey the field of application and to show how problems of free surface, friction and strain hardening can be incorporated in the solution process and third to extend the computation to thermally coupled problems where heat generation effects seriously enter the picture. Finally, in addition, the practical problem of elastic spring back is considered.

2. THE GENERAL CONSTITUTIVE RELATIONS AND THE BASIC FORMULATION

If elastic deformation is negligible, a very general description of behaviour of most materials can be given in terms of viscoplasticity. A particular form of this can be written following

†Reader in Civil Engineering, University of Roorkee, Roorkee, (U.P.) India.

Perzyna[12, 13] and defines the strain rate $\dot{\epsilon}_{ij}$, by a general expression

$$\dot{\epsilon}_{ij} = \gamma\langle\phi(F)\rangle\frac{\partial Q}{\partial\sigma_{ij}} \tag{1}$$

where

$$F = F(\sigma_{ij}, T, \epsilon_{ij}) \tag{2}$$

is the description of a yield surface

$$Q = Q(\sigma_{ij}, T, \epsilon_{ij}) \tag{3}$$

is a definition of a plastic potential.

Here

$$\begin{aligned} \langle\phi(F)\rangle &\equiv \phi(F) \quad \text{if } F \geq 0 \\ \langle\phi(F)\rangle &= 0 \qquad\ \ \text{if } F < 0 \end{aligned} \tag{4}$$

and both the yield and potential functions may depend on temperature T. The particular form of the function ϕ can be defined to fit in with experimental data.

In general Q and F will be different specifying thus a non-associated flow but invariably the relationship (1) can be written in a form

$$\dot{\epsilon}_{ij} = \Gamma_{ijkl}\sigma_{kl} \tag{5}$$

where

$$\Gamma = \Gamma(\sigma_{ij}, T, \epsilon_{ij})$$

is symmetric. Particular forms of above relationships are given explicitly in Ref. 13 for several yield and potential surfaces.

If comparison is made with the standard, linear, elasticity relationships

$$\epsilon_{ij} = C_{ijkl}\sigma_{kl} \tag{6}$$

it is immediately evident that the problem of pure visco-plastic flow (in which dynamic effects are ignored) and that of infinitesimal strain elasticity can be identified providing we replace

strains by strain rates
displacements by velocities

and allow the "elastic" constants C to be variables dependent on the stress or strain level.† This well known analogy is of immediate practical significance, as the possibility of solving flow problems by any numerical procedure capable of solving the equivalent elasticity forms becomes obvious. We thus do not have to enter here into details of the finite element formulation which

†In both elasticity and viscous flow the same equation of equilibrium holds, i.e.

$$\sigma_{ij,j} + b_j = 0$$

where b_j are body forces.

Further

$$\dot{\epsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where u_i is the velocity in the flow problem and identical expression is valid for small strain of

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where u_i is the displacement.

are used throughout the paper, and which follow standard well described patterns[14] beyond noting that an iteration will have to be done due to the dependence of $\mathbf{\Gamma}$ on the stress (or strain) levels.

In the discretised form we can define the whole problem as

$$\mathbf{Ka} = \mathbf{f} \quad \text{with } \mathbf{K} \equiv \mathbf{K}(\mathbf{a}) \tag{7}$$

in which $\mathbf{a}$ are the problem parameters (say nodal velocities), $\mathbf{K}$ appropriate "stiffness" matrices and $\mathbf{f}$ the system forces. The matrix $\mathbf{K}$ will now depend on the stress (or strain rate levels) and the solution will have to be iterative. The scheme adopted is simply that of successive updating, i.e.

$$\mathbf{a}_m = \mathbf{K}_{m-1}^{-1}\mathbf{f} \tag{8}$$

which is, if the problem is "well posed", converges very rapidly.

In the situations of forming discussed here invariably the problem has velocity conditions prescribed on some boundaries, and this indeed is the forcing effect. It is therefore important to be able to describe the flow matrix components ($\mathbf{\Gamma}$) in terms of *the strain rates* rather than in terms of the stresses. This will specify the problem in a "well posed" manner. Before proceeding further we shall specialise the general flow description of eqns (1) or (5) to the case of Von Mises type characteristics which are applicable to many metals and plastics.

3. VON MISES TYPE FLOW

The most common description of visco-plastic flow of metals (and many other materials) follows the assumption that both the yield and plastic potential surfaces are identified (i.e. the flow is associated) and that these depend only on the second stress strain invariant, i.e.

$$F = Q = \sqrt{3}\sqrt{J_2} - \sigma_y. \tag{9}$$

Here

$$J_2 = \frac{1}{2} s_{ij} s_{ij}$$

is the second invariant of the deviatoric stress components s_{ij} and σ_y is the uniaxial yield stress.

In general σ_y itself is dependent on the temperatures T and the accumulated effective strain $\bar{\epsilon}$. This second invariant can be obtained by time integration of the appropriate rate expression given by

$$\dot{\bar{\epsilon}}_{ij}^2 = 2\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}. \tag{10}$$

Some further generalisation can be obtained by making the yield stress σ_y depend on the mean stress (or pressure p). Thus if

$$\sigma_m = \sigma_{ii}/3 = -p \tag{11}$$

we can write

$$\sigma_y = \sigma_y(T, \bar{\epsilon}, \sigma_m). \tag{12}$$

In most problems the dependence of yield strength on the pressure is insignificant and is omitted in most of the latter examples. However, it presents a useful device for dealing with friction conditions when these arise. If we thus impose direct proportionality of σ_m and the yield strength in a certain zone writing

$$\sigma_y = -\eta\sigma_m \tag{13}$$

where η is the friction coeffcient the material so modelled will give a reasonable approximation to purely frictional behaviour near a surface where shear stresses are limited by the product of normal stress and friction coefficients.

Insertion of a narrow layer of elements with such a property near a boundary is thus a simple way of representing a boundary friction effect.

With Q written in terms of eqn (9) and assumed to be independent of strains and stresses and temperature we can write the general flow equation equivalent to eqn (5) as

$$\dot{\epsilon}_{ij} = \gamma\langle\phi(\sqrt{3}\sqrt{J_2} - \sigma_y)\rangle \frac{\sqrt{3}}{2\sqrt{J_2}} s_{ij}. \tag{14}$$

Clearly the strain rates implicit in above expression are such that

$$\dot{\epsilon}_{ii} = 0$$

i.e. the material flows without volume change and in the equivalent elastic model we have to assume incompressible behaviour.

We can also compare the expression (14) to a viscous isotropic shear deformation given by

$$\dot{\epsilon}_{ij} = \frac{1}{2\mu} s_{ij} \tag{15}$$

in which μ is the viscosity constant. (In the elastic analogies the shear modulus is thus equivalent to viscosity).

The viscosity is identified using eqn (14) as

$$\frac{1}{2\mu} \equiv \frac{\gamma\sqrt{3}}{2\sqrt{J_2}} \langle\phi(\sqrt{3}\sqrt{J_2} - \sigma_y)\rangle \tag{16}$$

and this is a function of the stress level.

To write an equivalent expression in terms of the strain rate level we note from eqn (15) and eqn (9) that

$$J_2 = 2\mu^2 \dot{\epsilon}_{ij}\dot{\epsilon}_{ij} = \mu^2 \dot{\bar{\epsilon}}^2. \tag{17}$$

Substituting above into eqn (16) we have

$$\dot{\bar{\epsilon}} = \gamma\sqrt{3}\langle\phi(\sqrt{3}\mu\dot{\bar{\epsilon}} - \sigma_y)\rangle \tag{18}$$

from which μ can be found for any strain rate function. A special but still quite general form of the function ϕ is an exponential type law and now eqn (14) takes the form

$$\dot{\epsilon}_{ij} = \gamma\langle(\sqrt{3}\sqrt{J_2} - \sigma_y)^n\rangle \frac{\sqrt{3}}{2\sqrt{J_2}} s_{ij}. \tag{19}$$

In this case we can evaluate μ explicitly from eqn (18) as

$$\mu = \frac{\sigma_y + (\dot{\bar{\epsilon}}/\gamma\sqrt{3})^{1/n}}{\sqrt{3}\dot{\bar{\epsilon}}}. \tag{20}$$

Expressions (19) and (20) are widely applicable. Ideally plastic materials with a fixed yield point are simply obtained by taking $\gamma = \infty$ and give

$$\mu = \frac{\sigma_y}{\sqrt{3}\dot{\bar{\epsilon}}}. \tag{21}$$

This value tends to infinity as $\dot{\epsilon}$ tends to zero so in numerical computation a large but finite cut off value must be assumed.

Such a cut off value defines in fact a finite viscosity everywhere thus allowing stresses to be computed even in zones where these are below the yield condition (rigid or nearly rigid behaviour). This model is used for most other subsequent computations.

A purely creeping material is characterised by $\sigma_y = 0$ and from eqn (19) we have in this case

$$\dot{\epsilon}_{ij} = \frac{\gamma 3^{(n-1)/2}}{2}(\sqrt{J_2})^{n-1} s_{ij}. \tag{22}$$

This expression is identifiable with that used by Cornfield and Johnson[8] where the explicit temperature dependence is specified.

$$\dot{\epsilon}_{ij} = A(\sqrt{J_2})^{n-1} s'_{ij}\, e^{(-Q/RT)}. \tag{23}$$

4. SOME NUMERICAL FORMULATIONS FOR VON MISES TYPE FLOW

The obvious characteristic of the Von Mises type flow described by eqn (14) and in specialised forms in eqns (19) and (20), is that the volumetric strain rates are zero. The corresponding elastic solution of the problem has thus to be capable of dealing with incompressibility.

Standard displacement formulation of the problem and its discretised forms apparently fail for incompressibility as this corresponds to an infinite value of the bulk elastic modulus (or to an infinite value of the Lame constant λ defining incompressible elasticity by the stress strain relation of the following form)

$$\sigma_{ij} = \lambda \epsilon_{ij} \delta_{ij} + 2\mu \epsilon_{ij}. \tag{24}$$

We have, therefore, to search for elasticity solutions of the type which permit incompressibility to be achieved. Many such formulations are possible but only two of particular merit have been investigated in detail in this paper.

(a) *The* **u**/*p formulation with Lagrangian constraint.* In this formulation the variables are the displacement or in flow case velocity variables **u** with an associated Lagrange multiplier variable which is identified as the mean stress (or simply the pressure). To achieve this formulation the potential energy is written in terms of deviatoric strains and is constrained to achieve incompressibility. This form was first introduced by Herrman[15] and elaborated in the context of fluid mechanics by many researchers[5]. This last reference gives the full details of matrix forms so that repetition is not necessary.

(b) *Penalty function approach.* Here the formulation adopts the pragmatic view and proceeds from the standard displacement form making the bulk modulus (or λ) have a large but not infinite value. This corresponds in elasticity to taking a Poissons ratio as a number approaching, but not equal, to 0.5. This formulation is precisely that achieved by again expressing the strain energy in terms of the deviatoric strains and imposing the incompressibility constraint by a penalty function method[16, 4].

Both approaches thus start from essentially the same point and differ only in the manner in which their constraint is imposed. For the success of *either* procedure it is essential to ensure that the discretisation is not over *constrained*, and that sufficient degrees of freedom are available to characterise a realistic displacement or velocity field when the constraints are imposed. It is the failure to observe this rule which makes the Herrman approach fail disastrously if linear/constant strain, triangles are used with piecewise constant parameters p applied to each element (thus imposing totally the incompressibility condition)[17]. For the same reason the linear constant strain triangles are inapplicable when quite moderate values of Poissons ratio (e.g. $\nu = 0.45$) are used.

In the present paper the **u**/p formulation uses an eight node isoparametric quadrilateral element to describe the velocities and in this a linear interpolation is used for the pressure to ensure sufficient freedom for the velocity variables[18] (Fig. 1e). In the penalty function approach the use of the nine node isoparametric Lagrangian quadrilateral element for velocity distribution has proved to give generally more accurate results than the eight node one, previously adopted for such situations [4, 5, 19], indicating its superiority for incompressibility situations, Fig. 1(b). In both nine and eight node elements the necessary relaxation of constraint is supplied by using "reduced" integration (2×2 Gauss). With this effective Poissons ratio of 0.4995 are readily achieved without numerical difficulty [20].

Clearly other types of elements could be used with success, and in particular, we would like to mention the very useful development of using a linear quadrilateral with reduced integration (single point) applied to the incompressibility condition. This element has been developed by Hughes *et al.* [21].

As we have already mentioned, other formulations for dealing with incompressible behaviour are possible. The most obvious is the use of a stream function variable, or of various equilibrium and hybrid forms with appropriate variational principles. We have abandoned the first possibility [3] as it is limited to two dimensional situations and, thus, not generally applicable, but many of the other alternatives can be effectively used and yet remain to be investigated.

In all approaches the essential procedure for dealing with plastic or viscoplastic flow processes follows a simple and well defined pattern which depends for its success on the strict analogy with elastic formulation. It is worth while at this stage to summarise its essentials as any computer program with the capability of dealing with incompressible (or near incompressible) elastic solutions can be readily adopted to solve the classes of problems discussed in the examples that follow. Thus we can always proceed as follows:

(1) Identify an isotropic linear elasticity program capable of solving incompressible problems. This program is formulated by supplying at the element level values of the shear modulus which is identified with the viscosity.

(2) As velocities and displacements are identified the first, trial, solution proceeds with appropriately prescribed boundary velocities and any arbitrary value of μ defining the viscosity at all points.

(3) The velocities obtained in solution of stage (2) define a strain rate for all elements (or appropriate element integrating points). Now expression such as eqn (20) is used to define a new set of viscosities applicable. At this stage temperature dependence or total strain dependence can be taken into account—with no additional difficulty.

(4) The new viscosities obtained in stage (3) are used over again to recompute the problem set in stage (1) with appropriate boundary conditions and a new set of velocities found. Successive repetition of stages (3) and (4) is carried out until full convergence is obtained. This can be judged by computing suitable error "norms" (see Ref. [14]).

The fact that the velocity distribution is relatively insensitive to the viscosity values guarantees fast convergence despite the fact that the "viscosities" may vary by several orders of magnitude.

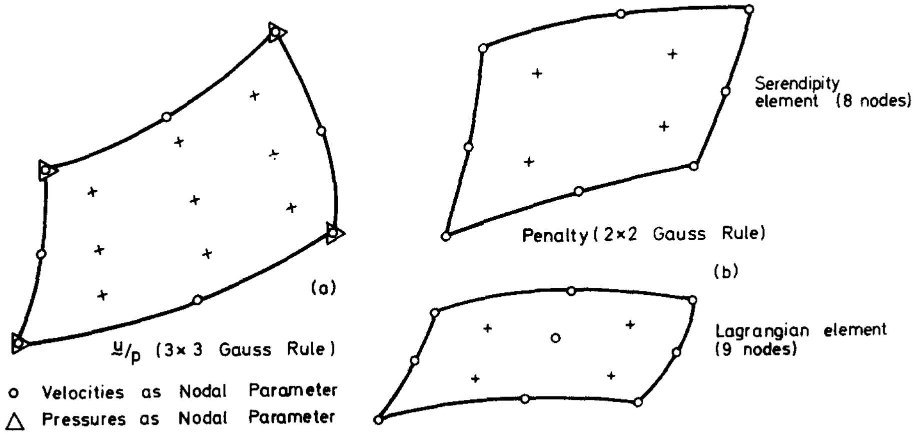


Fig. 1. Two dimensional **u**/p and penalty isoparametric parabolic elements.

When convergence is achieved the reactions, velocities and stress distribution, are readily available from the last, "elastic" solution obtained.

Clearly the simple iteration process typified by eqn (8) can be further accelerated by use of Newton–Raphson or other methods.

5. BOUNDARY CONDITIONS AND THE FREE SURFACE

By analogy with the elasticity problem, the *natural* boundary conditions of the corresponding flow formulations are tractions. The *forced* condition corresponds to prescribed velocities. Thus, in any problem in which a free (tractionless) surface exists, the velocity vector will be obtained on such boundaries as part of the solution.

Two distinct classes of problem now arise:

Transient problems

Here the knowledge of the surface velocity at any instant allows the surface to be updated by $\mathbf{u}_s\Delta_t$ (where $\mathbf{u}_s$ is the velocity vector) and a new solution to be found at time $t+\Delta t$ in the usual manner. Such updating is easily carried out also at all mesh points and new co-ordinates of this are then immediately available. However, on some occasions this may result in a grossly distorted mesh and it is then advisable to update the surface only and regenerate the internal mesh[3, 4].

Steady state problems

Here the requirement is that the final velocity is tangential to the free surface. This again can be carried out by an iterative updating as shown in Fig. 2(a) but usually convergence is difficult to obtain and an alternative approach is preferable. This is simple when on the free boundary a single point exists at which co-ordinates are known. Such a point may be defined, say, at the exit from a jet. From this fixed co-ordinate the whole surface can be determined by a simple integration starting from the fixed point as shown in Fig. 3.

Noting that at steady state

$$\frac{dy'}{dx'} \equiv \alpha \quad \text{or} \quad y' \equiv \int_0^{x'} \alpha \, dx'$$

where x' and y' refer to surface co-ordinates and where

$$\alpha = \tan^{-1} \frac{u_{ys}}{u_{xs}}$$

represents the velocity vector's direction, we can calculate surface co-ordinates corresponding to previously determined velocities. Successive recalculation of the free surface tends to a rapid convergence, and such techniques have been used in some of the jet problems which will be discussed.

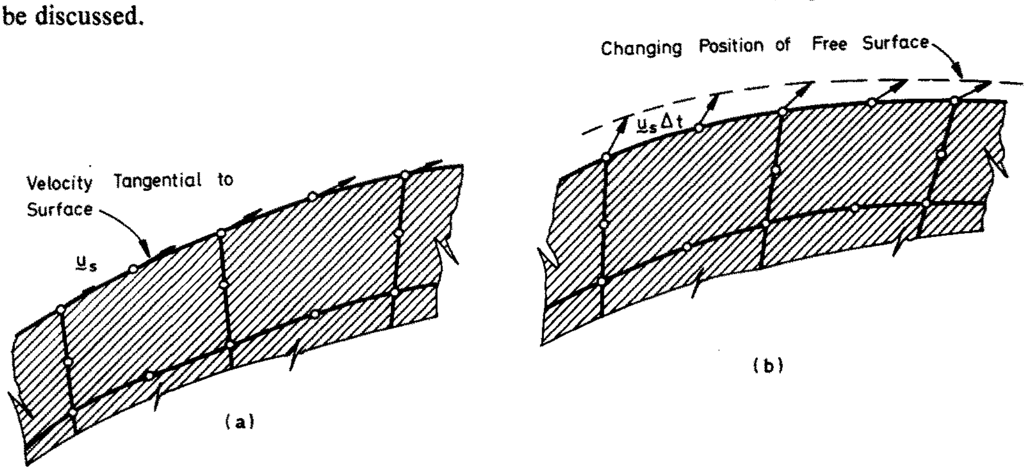


Fig. 2. Free surface; steady (a) and unsteady(b) types of solutions.

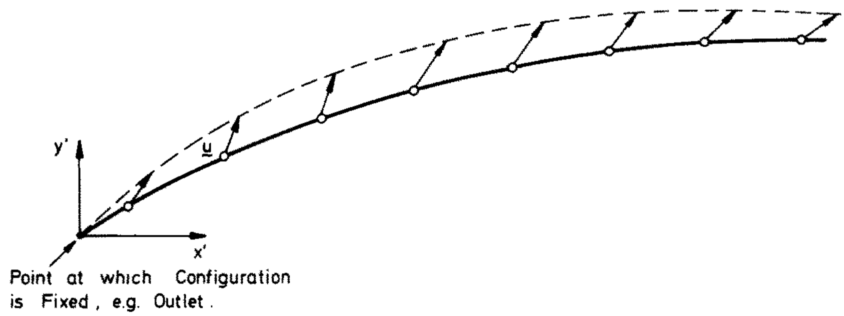


Fig. 3. Iterative approach to a free surface if a fixed point on the surface exists.

6. STEADY STATE FLOW SITUATIONS—SOME EXAMPLES

6.1. *Plane strain extrusion with slip boundary. Comparison of* **u**/*p and penalty methods*

This first example is similar to those treated in Refs. [3, 4, 11] and consists of a simple plane strain extrusion of an ideally plastic metal through a die with a 1:2 reduction. The walls of the container are assumed frictionless and the "jet" to issue in a parallel stream, this being, indeed, found to be correct on applying the free surface condition.

The reason for including this simple problem is that here analytical slip line solution exists[1], and comparison of the two discretisations is readily made. Constant yield stress value is assumed, and the thermal effects and strain hardening are excluded.

In Fig. 4 the general geometry of the problem is shown and also the two meshes used in the solution. The same figure shows the improvement realised by making the penalty function mesh singular at the exit corner.

The results of this study showing the comparisons of the **u**/*p* and penalty methods indicate reasonable predictions of collapse load even though the finite element discretisation does not permit velocity discontinuities to develop although these exist in the full theoretical solution. Despite this velocity "smearing" present, near such discontinuities, the velocity distribution checks reasonably with that of the slip line predictions as shown in Fig. 4(b).

It is important to note that results obtained using the penalty method with the nine node isoparametric quadrilateral element approximate, even for a coarse mesh, the exact solution much better than those obtained with the eight node element and the **u**/*p* solution for the same mesh. This is significant because it makes the penalty function approach a competitive method for this kind of problem against the **u**/*p* solution which involves a greater number of variables and hence a greater cost.

6.2. *Plane strain extrusion-friction behaviour*

If the boundary of the container is rough and friction is present, the forces necessary for extrusion increase with the length of the section between the piston and the die. For this reason, the solutions are less general than those for the frictionless case, and few results are quoted in literature. In Fig. 5 we show the application of the procedures to a particular case of plane strain extrusion with a given piston/die configuration. Here the effects of boundary frictions were introduced by inserting a narrow layer of elements between the boundary and the flow and assigning different values to the friction coefficient using the process suggested in Section 3. Results for fully rough boundaries, i.e. with no slip allowed, and those with a friction coefficient $\eta = 0.1$ and finally for a full slip condition are shown in Fig. 5. As expected, these follow the right order.

6.3. *Axisymmetric extrusion of pipes and hollow sections*

Figure 6(a) and 6(b) show examples of an extrusion of an axisymmetric billet to form a hollow pipe. The problem here is treated as a semi transient one showing the various stages and positions of the extruded material. No friction is assumed in the above examples, and these are shown merely to indicate the practical possibilities of following real situations. Results obtained using the penalty approach with the nine node element are again very accurate for all stages and comparison with those obtained using **u**/*p* and with the experimental solution is excellent. On

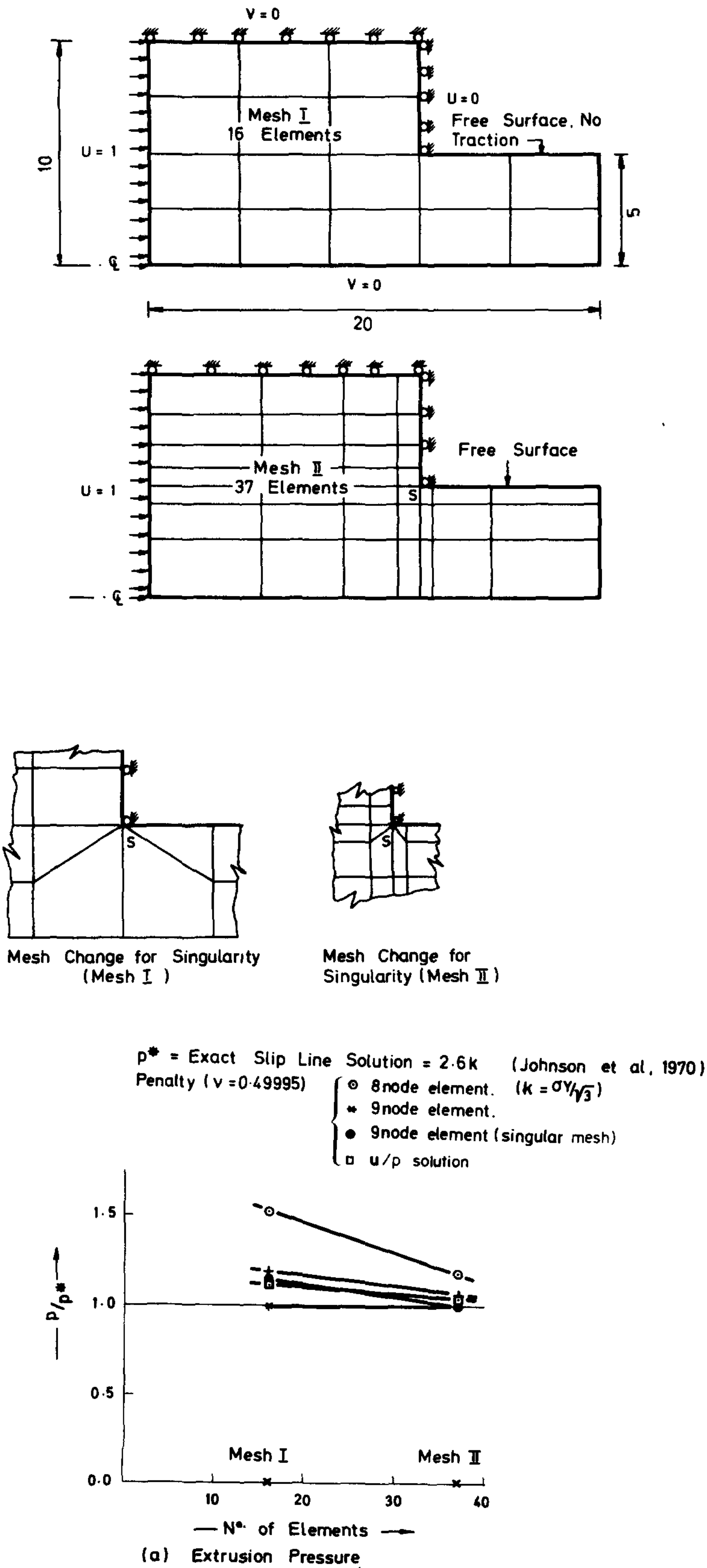
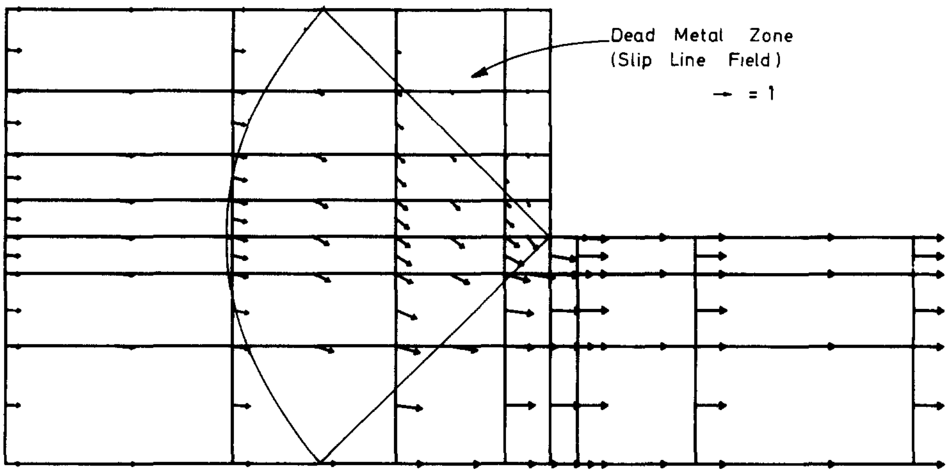
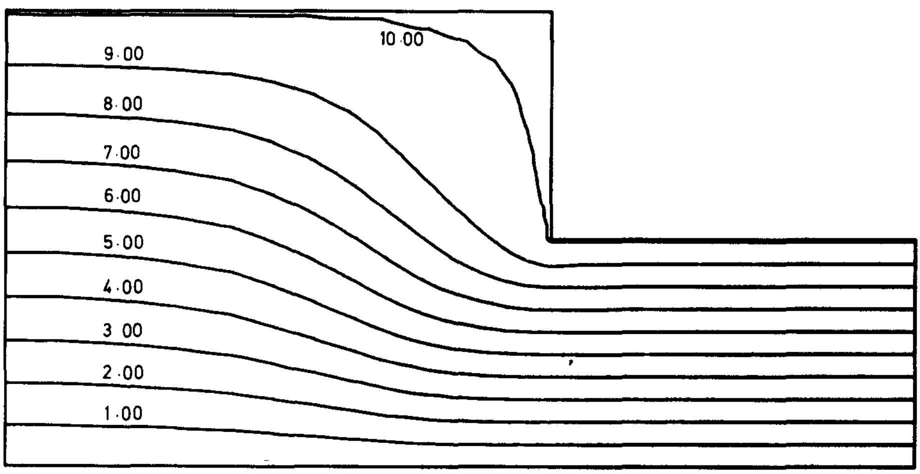


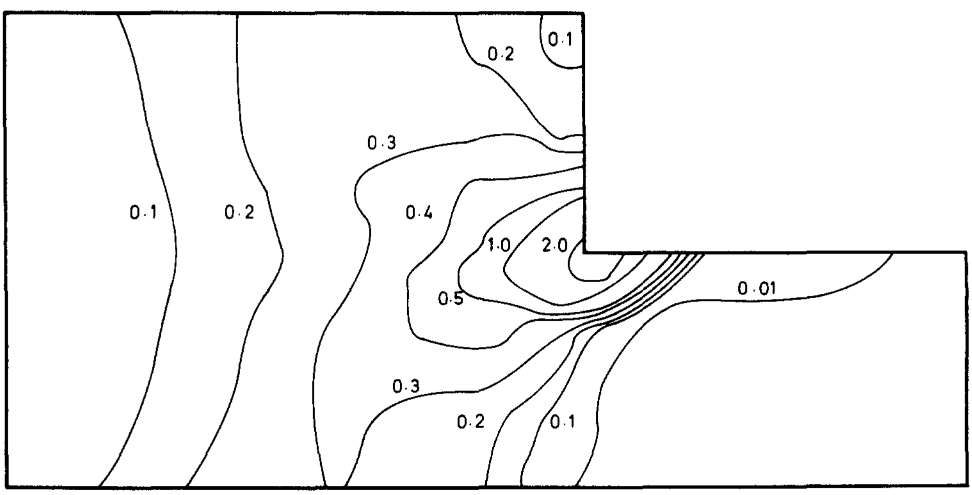
Fig. 4. Plane strain extrusion (reduction 0.5). Study of convergence to u/p and penalty.



(b) FINE MESH, VELOCITY VECTORS (37 ELEMENT, $\underset{\sim}{u}/p$)

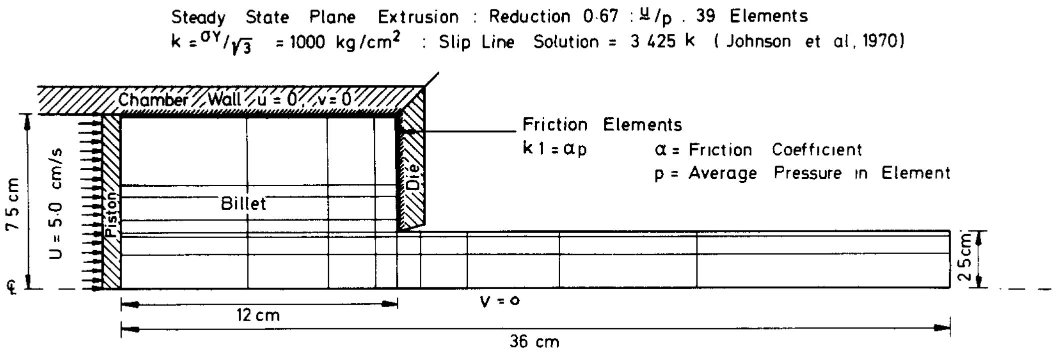


(c) FINE MESH, STREAM LINES (37 ELEMENT, $\underset{\sim}{u}/p$)



(d) FINE MESH, EFFECTIVE STRAIN RATE ($\dot{\bar{\varepsilon}}$) CONTOURS ($\underset{\sim}{u}/p$)

Fig. 4. (*continued*). Plane strain extrusion. Velocity vectors, stream lines and effective strain rate ($\dot{\bar{\varepsilon}}$) contours.



Boundary Condition	Extrusion Pressure
No Slip	4·537 k
α = 0·1	4 186 k
Full Slip	3·6998 k

Fig. 5. Plane extrusion: rough or friction boundary.

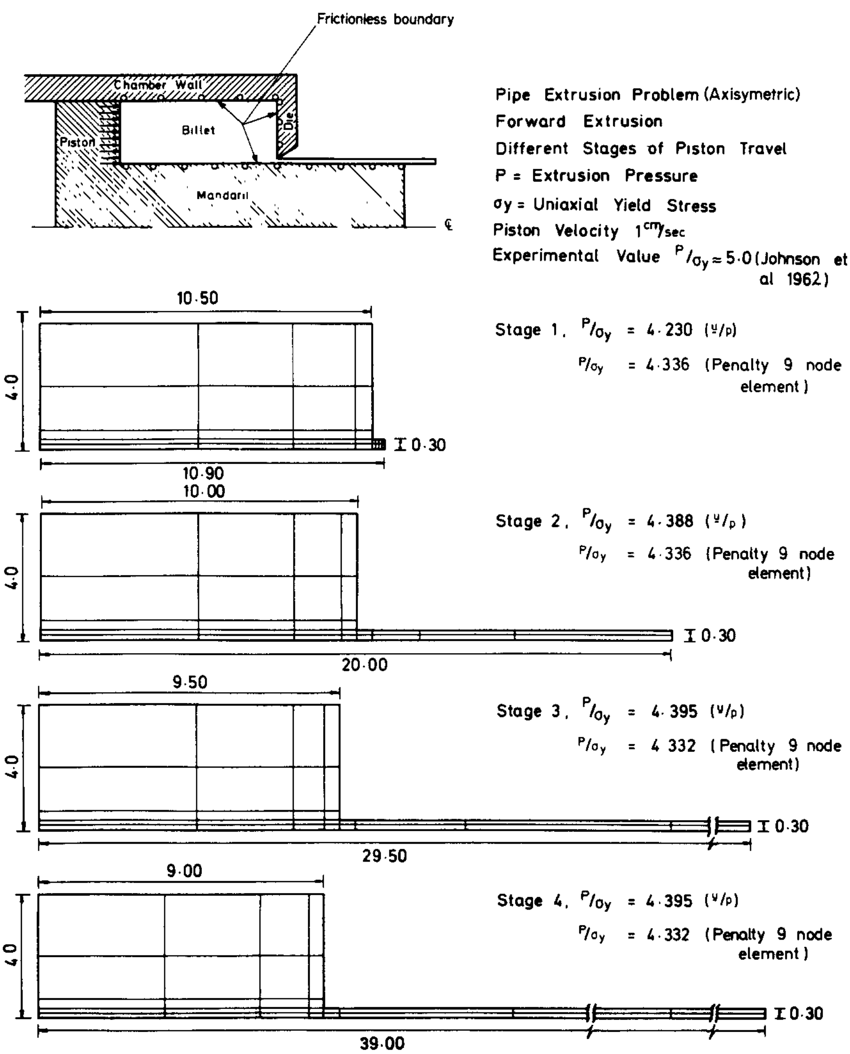
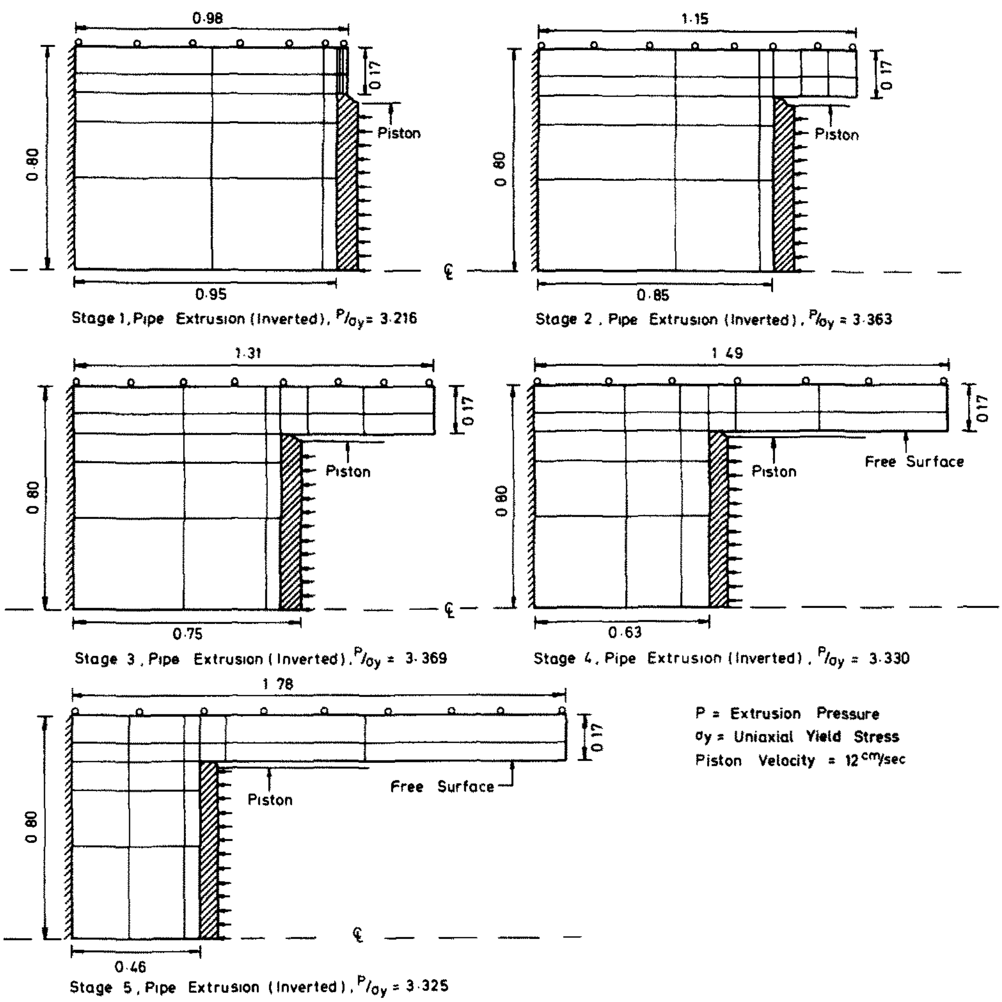


Fig. 6(a).



Pipe Extrusion Problem : Inverted Extrusion

Fig. 6(b). Axisymmetric pipe extrusion problem: inverted extrusion. Different stages of piston travel: extrusion pressures, ($\mathbf{u}/p$) solution.

the contrary results obtained with the penalty method using the eight node element have proved to be considerably inferior.

6.4. *Axisymmetric and plane jet problems*

Two problems of viscous jet flow issuing from a pipe or slit where laminar flow conditions with parabolic velocity are established are indicated in this section. In the pipe a fully rough boundary is assumed for such flow and the results are shown in Figs. 7 and 8 for a constant viscosity.

In the first problem of Fig. 7 the axisymmetric jet issues freely with no traction applied to the extreme end of it. In the problem of Fig. 8 prescribed velocity (or corresponding forces) are applied at the end of the jet drawing this out. In both problems constant viscosity is assumed, but obviously non-Newtonian characteristics could easily be taken. The problem has much application in glass industry where constant viscosity conditions are observed. The first problem has been previously solved by Nickell[22], and similar results are here again obtained. The second is of some practical applications and has only, to date, been examined by approximate one dimensional solutions[23] which here is used for comparison. Obviously the extension of such solution to non-Newtonian; viscoplastic materials does not present any additional difficulty[24].

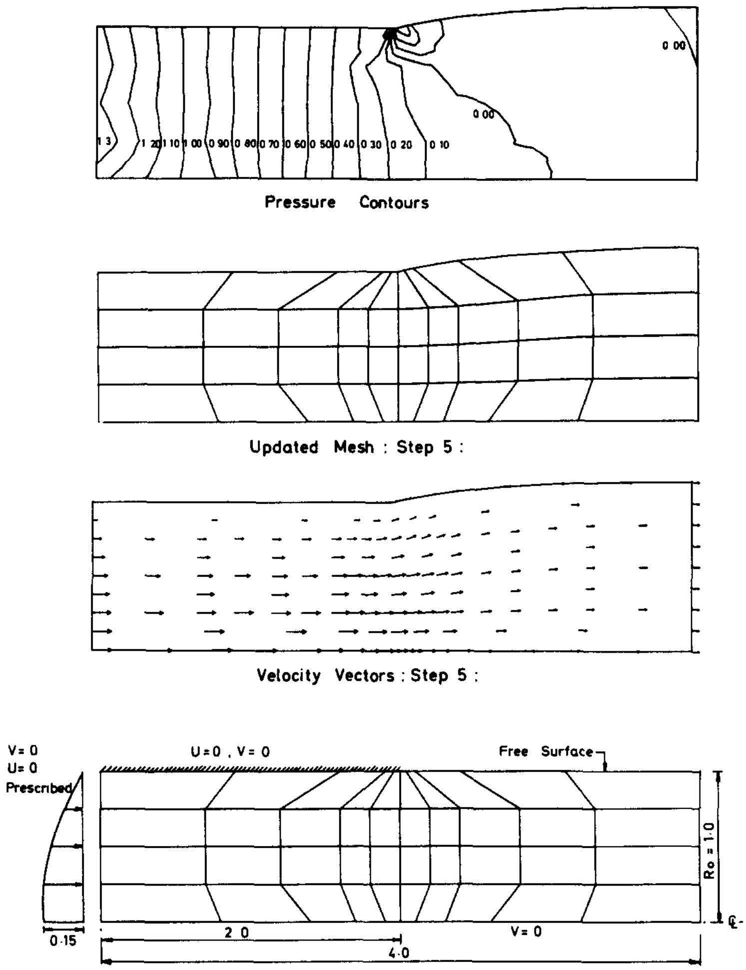


Fig. 7(a). Viscous incompressible jet expansion (axisymmetric) (u/p) solution.

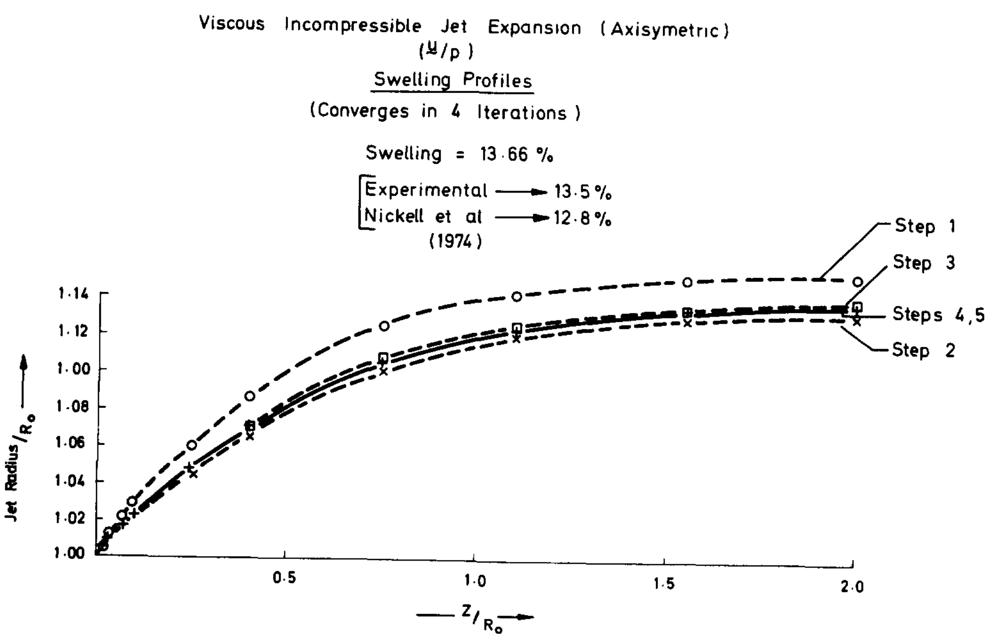


Fig. 7(b).

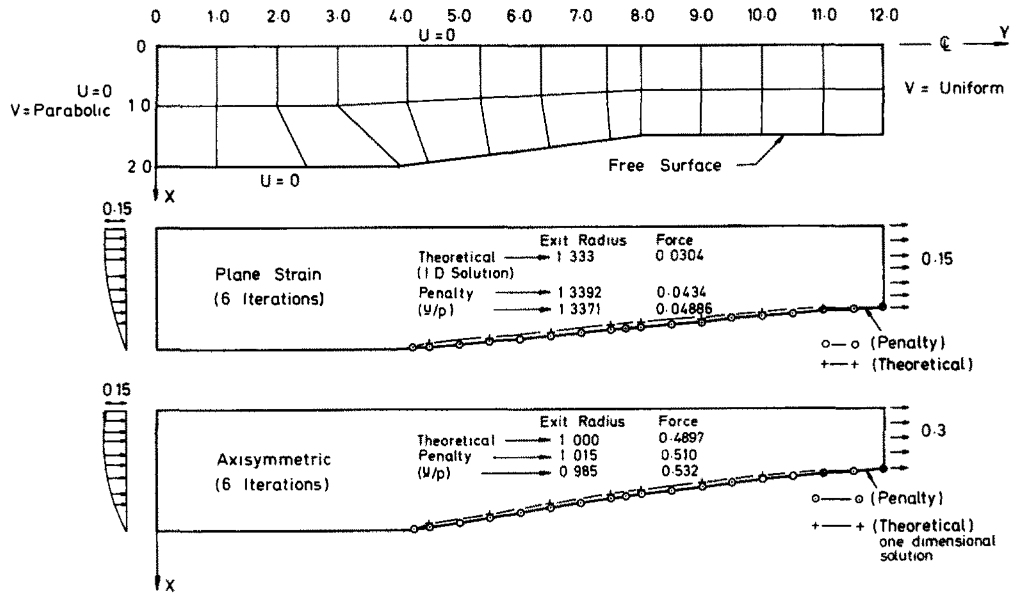


Fig. 8. Viscous incompressible jet-drawing problem.

6.5. *Rolling problems*

The process of rolling is, obviously, one of fundamental importance in metal forming, and two examples are included here to show the possible application of the procedures discussed. Obviously, in general, the problem is not one of steady state flow, but if the sheet rolled is of long extent it can be approximated as such.

Several difficulties are immediately encountered when the problem of rolling is considered. First, the position of the free surface which at the entry and at the exit from the roller is not uniquely defined. It is generally assumed that the inlet point is determined by the thickness of the slab entering and the exit occurs at the lowest section of the roll if no "sticking" occurs. To confirm, or deny, such assumption the first example of Fig. 9 is considered. Here, the figure shows the mesh used and the original position of a free surface at the upstream side corresponding with the thickness of the billet entering and the exit fixed arbitrarily at a certain point. In this example no slip is assumed to occur between the roll and the metal, i.e. imitating fully rough conditions of the roll. The analysis indicates, immediately, two interesting phenomena. The free surface profile ahead of the roller is changed for the given point of contact of the metal with the roll at entry. The manner of this change, which is not of great magnitude, is indicated in the figure and, as it were, denotes the up-flow spread of the rolling effect.

Of more importance is the assumption of complete sticking between the roll and the arbitrary exit point. The solution obtained with the configuration shown indicates a large tensile region, (with negative pressures) occuring downstream of the narrowest section between the rolls. If the contact with the rolls is such that it is incapable of carrying such tensile stresses the material must, obviously, separate from the roll at the earlier point, the calculation indicates that the point of separation occurs close to the narrowest section as, indeed, the practitioner would expect.

Having determined these effects we now are ready to turn to another realistic roll problem for which some results are available in literature Refs. [25, 26]. Now the mesh shown is indicated in Fig. 10(a) and this assumes the correct exit point. In this example two different boundary conditions are taken. First, a no slip condition, as before, is introduced, and this results in the velocity and surface pressure variation shown in that same figure. The numerical computation readily allows the total torque and forces on the roll to be computed and these are shown in a table. As the computation does not here take into account roll deformation no direct comparisons with solutions of Ref. [25] and [26] are given.

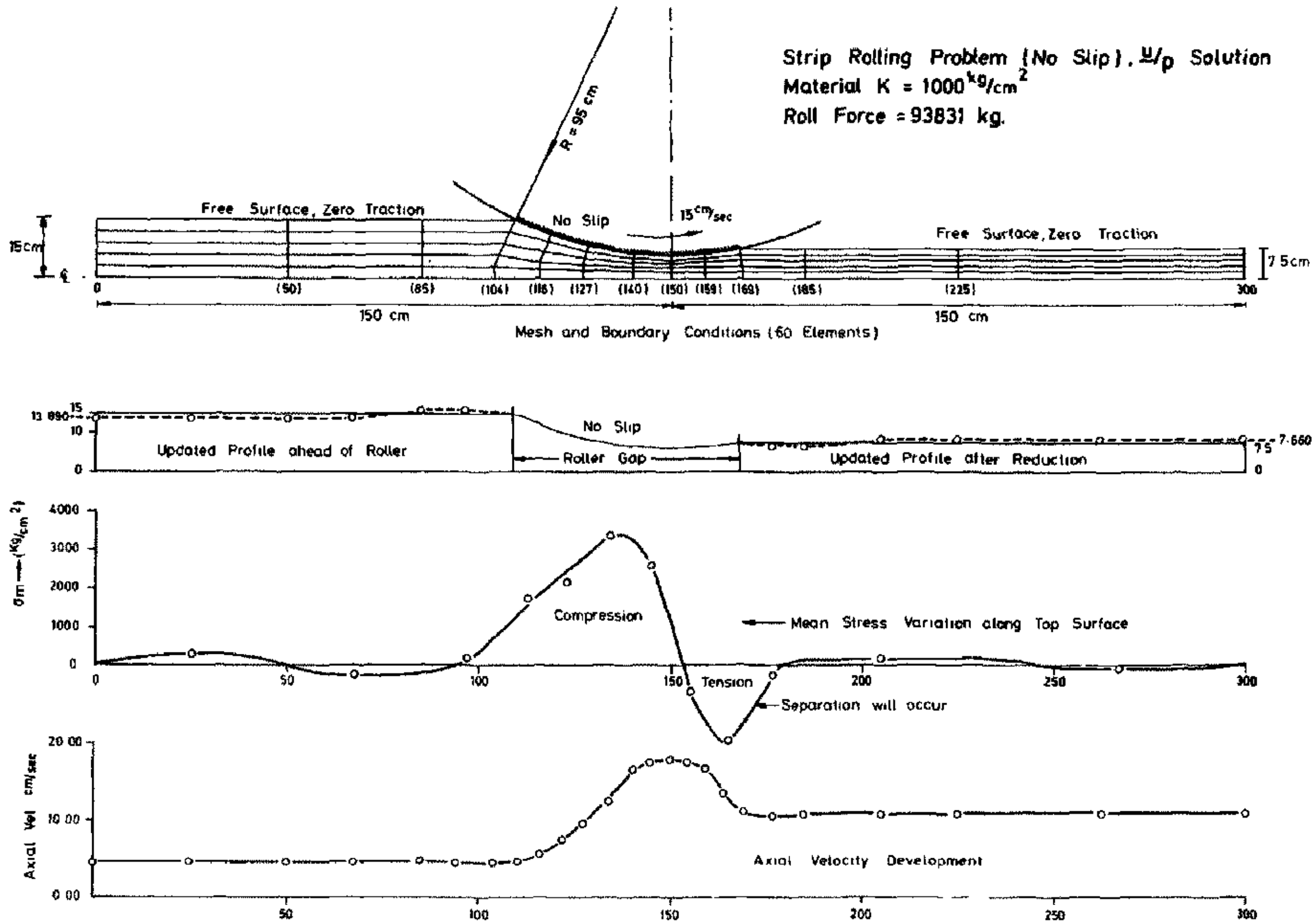
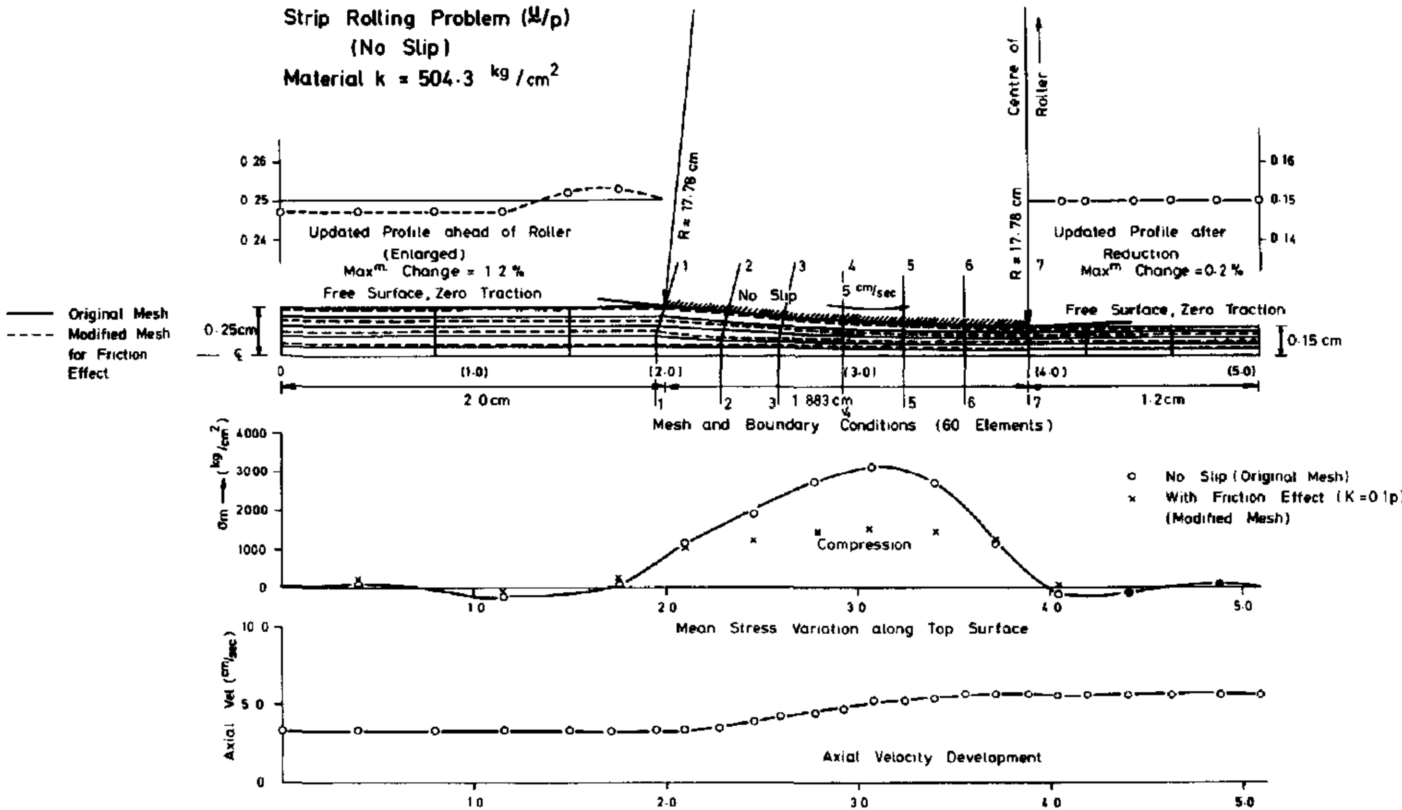


Fig. 9. Study of separation point.



	No Slip, Constant K (Y/p) 21 El	60 El	With Friction Effect
Rolling Force	4841·3 kg	4286·6 kg	2809·3 kg
Torque	4644·2 kg cm/cm	4013·9 kg cm/cm	2620·2 kg cm/cm

Fig. 10(a). Study of problem as analysed by Ford/Alexander (1972).

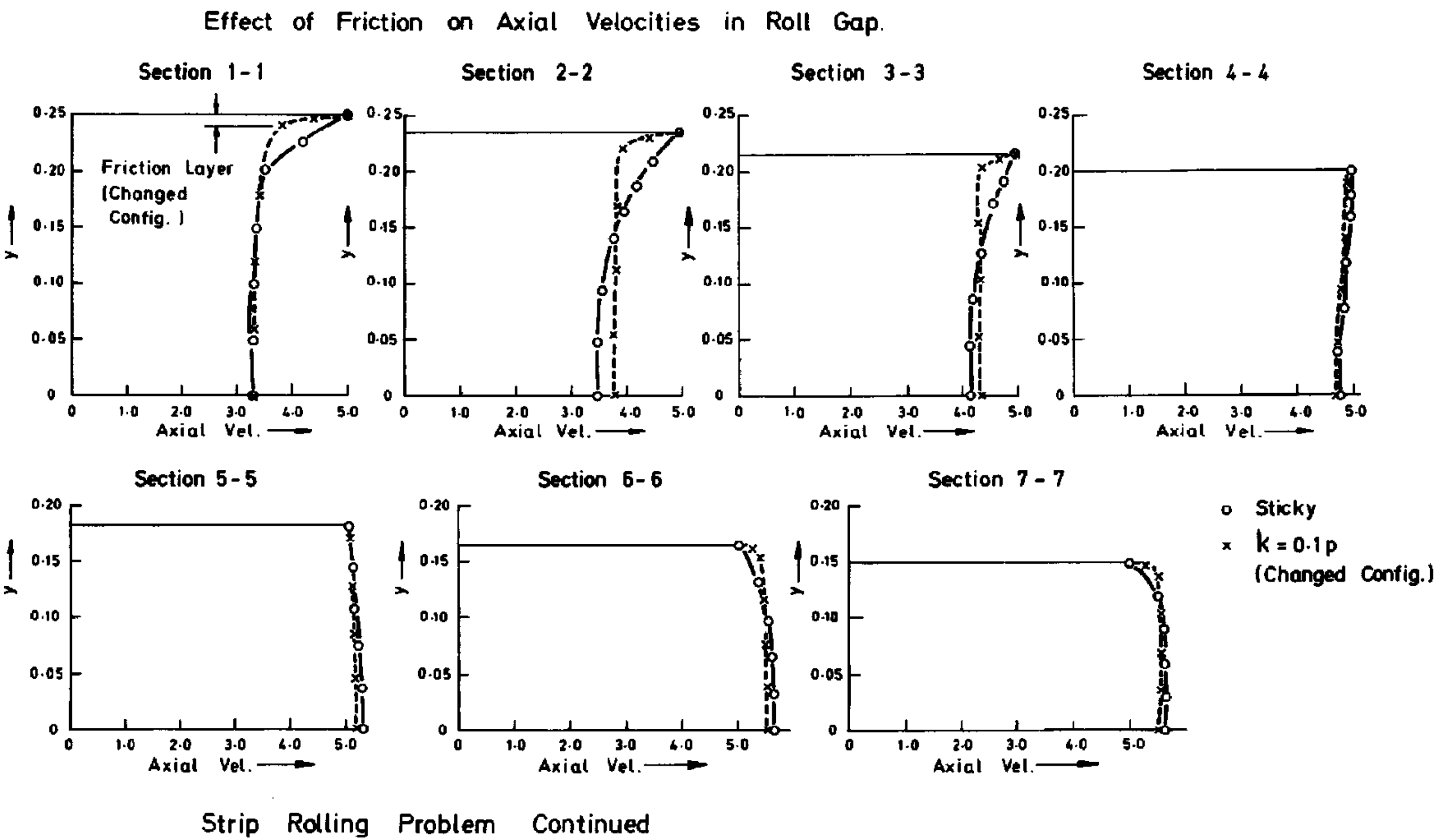

Effect of Friction on Axial Velocities in Roll Gap.
Section 1-1
Section 2-2
Section 3-3
Section 4-4
Section 5-5
Section 6-6
Section 7-7
Friction Layer
(Changed
Config.)
y
Axial Vel.
o Sticky
x k = 0·1p
(Changed Config.)
Strip Rolling Problem Continued

Fig. 10(b).

To investigate friction conditions near the surface the mesh near the contact points was altered so as to introduce a layer of friction type elements of the kind already discussed. The results with a friction coefficient $\eta = 0.1$ are indicated in the same table, and in Fig. 10(b) we show the velocity distribution across the various sections of flow for both no slip and frictional conditions. This follows a pattern which, obviously, could be explained physically.

It is clear that much more remains to be done in the rolling problem to take into effect such things as roll deformation, dissymmetry and three dimensional behaviour. However, the solution indicates that all of these aims are readily obtainable with a numerical finite element discretisation.

7. TRANSIENT PROBLEMS

7.1. *General*

Previous publications have already contained some transient type applications in which the surface is successfully updated as shown in Fig. 2. As each flow solution starts in an Eulerian manner from the present configuration very gross deformations can be readily followed by a simple process of repetition of the solution in updated co-ordinates. The field of application is obviously vast, ranging from that of punch indentation problems through forming of threads by impression to the problems of sheet metal deformation of the kind important in the motor car and similar industries. In the examples treated previously the simplification of constant contact points of the indenting tool and of the deforming metal have been present. The case of much wider interest is that in which progressively larger parts of the die come into contact with increasing deformation. It is with such a problem that we are here concerned.

7.2. *Deep drawing and stretching*

All the characteristic of a sheet metal forming process are present in the simple sphere indentation test used by the industry. Figure 11 shows a typical configuration of such a problem and of the boundary conditions assumed. If the metal in the original blank is held rigidly and not allowed to slip between the boundary, clearly it will thin out with progressive displacements of the spherical die and this type of situation is characterised as that of stretch forming. The other extreme in which the metal is allowed to slide freely between the holder is one of deep drawing. Most practical tests lie between the two extremes, and in the Fig. 11 we show how the results differ for the total force/versus stroke in both cases.

The illustration of progressive deformations occurring is that corresponding to the stretch forming problem and shows the limited number of elements used and the successive changes of the contact zone. In this example we have assumed a completely rough die such that once a point enters contact with the material no further slip occurs. Obviously this assumption could be modified by including friction type elements of the type already discussed, but it serves well our purpose of illustration and comparison. The computational procedure is of some interest and will be briefly mentioned. At the start of the problem we assume that only a single point at the mesh is in contact with the die, i.e. the point at the centre. Once the velocity field is determined for this configuration it is an easy matter to calculate by pure geometrical consideration the time interval necessary for the second mesh point to come into contact with the die. In general it will be found that this time interval corresponds to quite large deformations and to avoid inaccuracy this is divided into several increments with the point of contact assumed unchanged for all except the last time step at which the second point enters into contact. At that time the boundary conditions are appropriately changed and similar procedure is extended to achieve the next mesh point contact.

An obvious defect of this procedure occurs near the peak load where the stiffening due to the extending contact zone is not well modelled. An alternative which has been used also with success is to adjust the element mesh at each time step so as to avoid elements in partial contact—however if great accuracy is required use of small elements is advocated.

Similar difficulties are encountered at the support zone in the deep drawing problem where the simple translation of the mesh alters the contact point. Again the use of small elements or a successive remeshing is necessary.

The second difficulty which is liable to arise in more general situations is the possibility of contact breaking during the loading sequence. This could be readily taken into account by

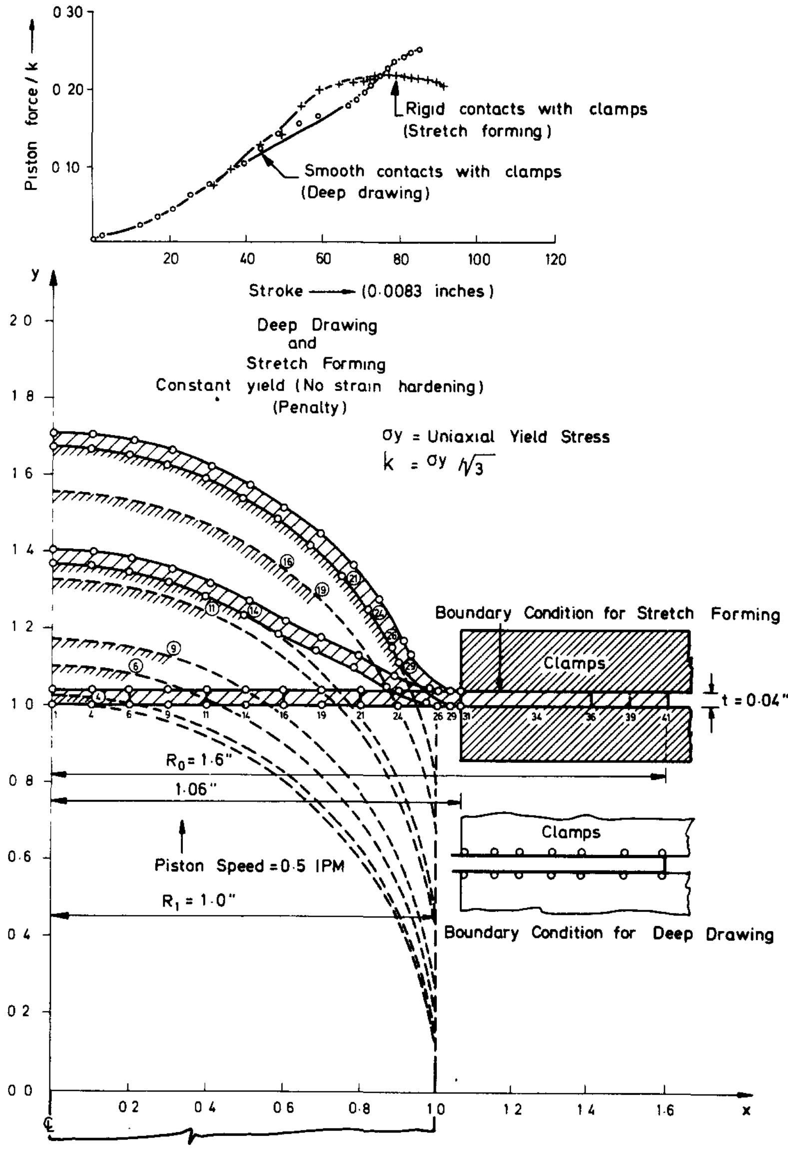


Fig. 11.

establishing the nature of the reactive forces. In the example computed this separation did not occur.

Figure 11 shows the results for both stretch forming and deep drawing in the case considered for which the yield stress has been taken as constant. These indicate the rapid drop of maximum force observable on experiments which occurs in the stretch forming case.

8. STRAIN HARDENING PHENOMENA

In all the examples considered so far we have assumed the material to be ideally plastic. Neither viscoplastic (time) effects have been considered, nor the change of yield strength with the strain. The true characteristics of most materials show considerable strain hardening as for instance, indicated in Fig. 12 for a certain alloy. In a transient situation it is not difficult to include such strain hardening in the calculation and, indeed, very little additional cost is involved. As at each stage of the deformation the strain rates are known, the total strain, and in particular the total (scalar) *effective second strain invariant* of eqn (10), can be evaluated, and the yield strength appropriately updated. To illustrate such effects two cases are considered. One refers to the stretch forming problems of the previous section. Here the material characteristics have been modelled by the curve shown in Fig. 12(a) and

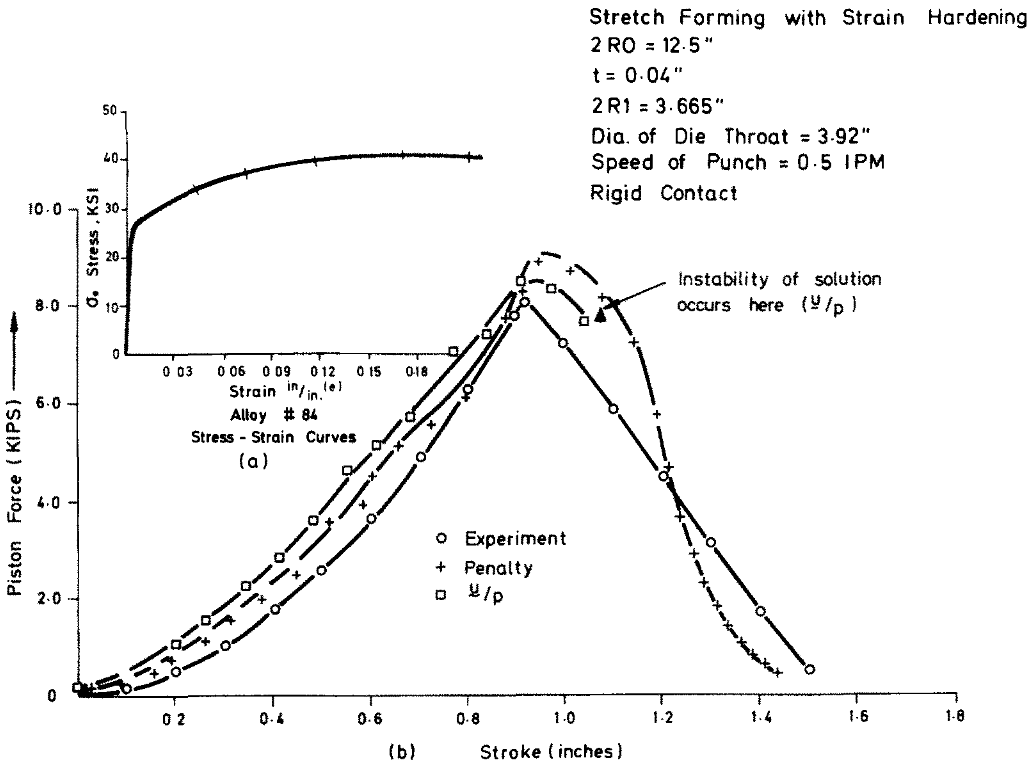


Fig. 12.

for which experimental results were available. The computation was carried out in exactly the same form as in example of Fig. 11 and the solutions are indicated in Fig. 12(b). Both penalty function and **u**/p approaches have been used here, and comparison between the two for the coarse mesh shows some discrepancies which are, of the same order as the error between experiment and theory. Some instability has been noted in the **u**/p formulation near the peak and this is not so clearly defined as with the penalty approach. Nevertheless, comparison with experiment is quite reasonable, and with more refined data, as well as meshes, we hope in the future to reach greater accuracy.

An additional example in this section refers to a problem similar to that discussed in Section 6.3 where a billet is subjected to an inverted extrusion process. The effects of strain hardening are considered and the results are compared with experimental results available[27] as shown in Fig. 13.

9. COUPLED THERMAL FLOW

In all the examples indicated so far isothermal conditions have been assumed which clearly do not correspond with practical problems where, very often, temperature changes occur either due to an external imposition of heat, or due to the spontaneous heat generation following the energy dissipation of the process. The extension of the process to prescribed temperature conditions is trivial. Clearly if the temperature is known *a priori* and the relation of the yield stress to it is given then no additional difficulties are introduced in any examples of the type considered here. This, indeed, has been done by Cornfield and Johnson[8] quite early on in the development of numerical solutions and extended to the process of dieless extrusion by Price and Alexander[10]. Of more complexity is the problem of determining the effects due to the energy dissipation occurring in the plastic process. Here the temperature development depends on the energy dissipation which in turn is dependent on plasticity values. The problem is then highly coupled. During any plastic or viscous deformation work is being dissipated at the rate equal to

$$Q = s_{ij}\dot{\epsilon}_{ij}. \tag{25}$$

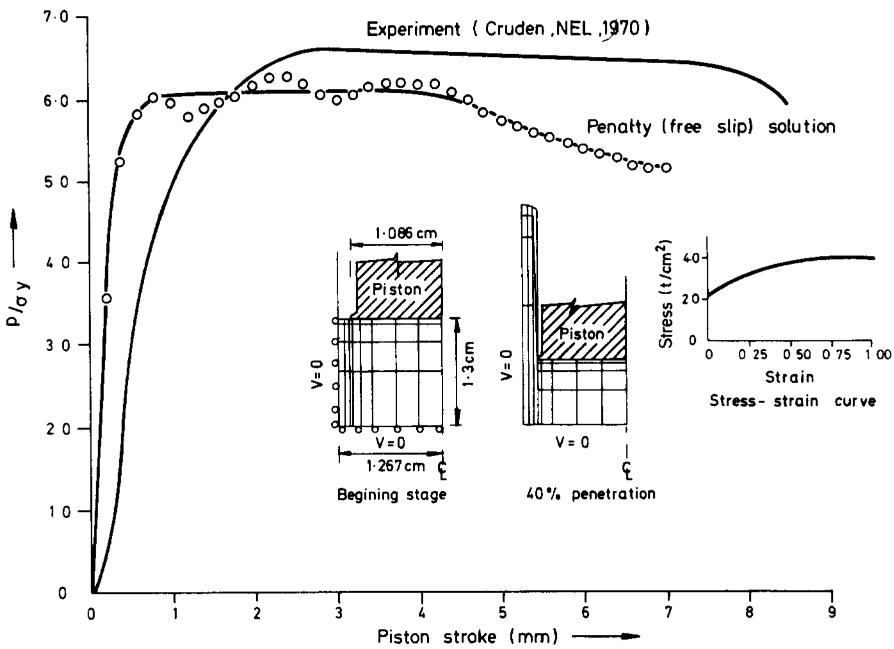


Fig. 13. Axisymmetric cup forming, inverted extrusion (with strain hardening).

This energy dissipation may be quite large and must be considered in the thermal equilibrium equations used for solving the temperature distribution problem. These are in plane flow

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+Q+pc\frac{\partial T}{\partial t}=pc\left(u\frac{\partial T}{\partial x}+\sigma\frac{\partial T}{\partial y}\right) \tag{26}$$

where k is the thermal conductivity and c is the specific heat.

The numerical solution of the diffusion convection equation is discussed in standard finite element texts[14] but certain aspects of the convective terms have only recently been satisfactorily solved. In particular the upwinding difference technique is necessary to ensure stability and formulations of the type given by Christie *et al.*[28] and Heinrich *et al.*[29, 30] are essential for the type of problem discussed. We shall not go into the detail of this type of computation but it is easy to see that an iterative procedure will have to be adopted to obtain a coupled thermal plastic deformation problem. The procedure adopted follows the lines of: (1) Solving the flow problem for a given temperature distribution, say assumed ambient, followed by; (2) solution of the thermal equations from which temperatures are calculated; (3) repetition of the solution of the flow problem with the plasticity values adjusted according to the temperature field, etc.

To date no such solutions have been reported in literature but, an uncoupled problem of heat generation has been solved by Bishop[29]. In the example of Fig. 14 we consider an extrusion problem somewhat similar to that discussed in the above reference showing a comparison of temperature distribution obtained using the upwinding technique of Refs. [29, 30] with the result of simplified computation made by Bishop[30]. In the same figure we show also the completely misleading results obtained by a simple application of the Galerkin—finite element procedure without upwinding to warn some of the readers of the possible difficulties.

The fully coupled solution is readily obtainable and the convergence is relatively rapid providing thermal properties do not change too rapidly.

10. ELASTIC SPRING BACK AND RESIDUAL STRESSES

So far elasticity effects have been explicitly ignored. Clearly these are of minor consequence providing the plastic deformations and flow is occurring. However, if at a certain stage of deformation the loads are removed, an elastic deformation occurs, and this is of some practical

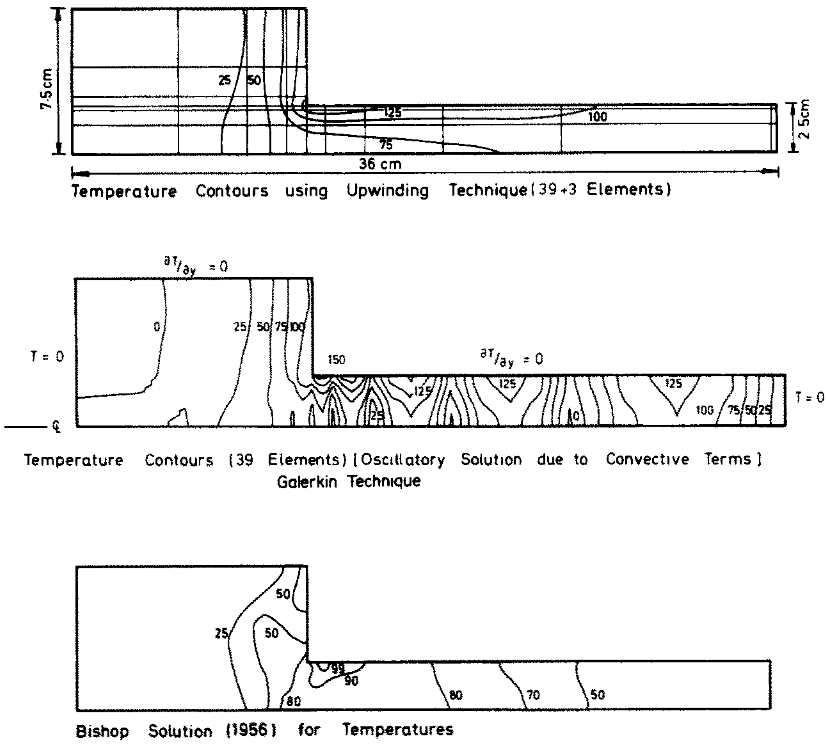


Fig. 14. Thermal problem of extrusion.

significance as the extruded or formed shape does not coincide with that desired. Further on such load removal residual stresses are developed.

The full solution of the elastic spring back problem is by no means simple but approximate results can be quite easily obtained if we consider the simple fact that on removal of the loads causing deformations purely elastic strains will occur. If we thus remove the reactive forces supplied by the forming tool the elastic spring back can be simply calculated by returning to the behaviour of a now purely elastic structure. Here an application of negative reactive forces on the deformed specimen provides the geometry of the analysis. Clearly a check should be instituted that all the deformations which occur are confined to the elastic zone, i.e. that no subsequent plasticity occurs and if a small displacement elasticity program is used, that this is applicable. In a general case, large deformation elastoplastic computation could be used to find the spring back results and, indeed, on some occasions this might be necessary as buckling and other phenomena can well happen, as is known in practice.

To illustrate a typical solution let us consider the example of Fig. 12 deformed to an intermediate stage by stretch forming and subsequent removal of the die. The forces applied by the die on the structure are shown on Fig. 15 and the deformations, now of small elastic kind, are computed taking the appropriate elastic modulus of the material as shown on the same figure. These have been obtained using a simple standard elasticity program.

The same unloading process can determine the residual stresses and these are shown in the same figure. A calculation of such residual stresses is of considerable importance in some forming processes and quite generally can be adequately modelled by the combination of the elastic unloading superimposed on the plastic forming stresses.

The simplified approach to the spring back process given here is by no means conclusive and much further experimentation is necessary to determine its applicability in practical cases.

11. CONCLUDING REMARKS

In this survey of possibilities and problems encountered in plastic deformation occurring in the forming of metal and other materials we have indicated the practicability of solving a wide range of situations. Clearly the configurations and details of application will be elaborated and

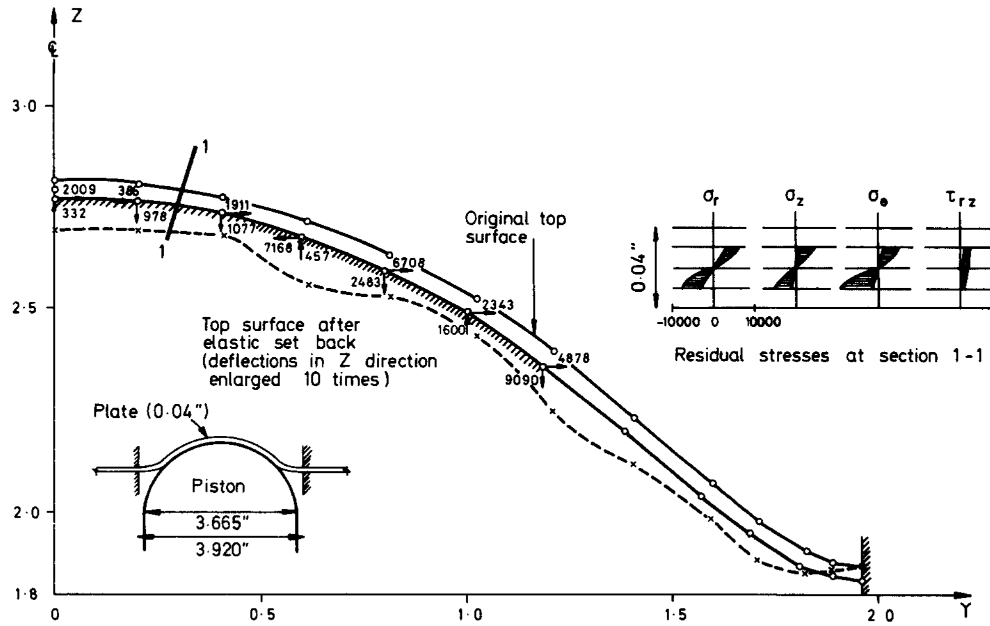


Fig. 15. Elastic spring-back in stretch forming problem.

made more complex in realistic examples but principles of a wide range of solutions are here included. One aspect which is now receiving attention and which should result in a more economical and practical treatment of large, thin sheets subject to pressing includes membrane/shell formulations in place of two or three dimensional analyses. The examples of Fig. 11 and 12 are typical here, and considerable computational economy could have been obtained by one dimensional rather than two dimensional modelling. Clearly application of elements of a fully three dimensional form would be prohibitively expensive in study of deformation of a large sheet to an arbitrary three dimensional form and simplification is essential.

Another problem of practical importance is that of three dimensional extrusion where often the specimen does not precisely follow the form of the roller. Here elaboration of such details as contact point separation discussed in Section 7 and of the appropriate friction considerations will have to be given considerable attention.

Finally we should draw attention to a very practical problem which appears to occur in extrusion of three dimensional shapes. Here, as the practitioner well knows, the die is often formed of sheets with "exits" of different thicknesses so as to ensure that the resulting section proceeds in a straight line. To date the experience of the die maker has been the primary motivation for the design of such dies who assigns appropriate thicknesses to each section of the extruded form. Computations of the type indicated here could well help in this design process and given frictional conditions would determine the shape of the form which leads to a straight extrusion. Much work remains yet to be done on this and other aspects of the problem, but the possibilities indicated here could form a basis for further studies.

Acknowledgements—The authors would like to thank the ALCOA Foundation of U.S.A. for a Research Award which has enabled some of this work to proceed; also ALCOA is to be thanked for the supply of details of experimental results for Fig. 12 of this paper.

REFERENCES

1. W. Johnson, R. Sowerby and J. B. Haddow, *Plane Strain Slip Line Fields: Theory and Bibliography.* Arnold, London (1970).
2. G. Y. Goon, P. I. Poluchin, W. P. Poluchin and B. A. Prudcowsky, *The Plastic Deformation of Metals,* Metallurgica, Moscow (Russian) (1968).
3. O. C. Zienkiewicz and P. N. Godbole, Flow of plastic and viscoplastic solids with special reference to extrusion and forming processes. *Int. J. Num. Meth. in Engng* **8**, 3–16 (1974).
4. O. C. Zienkiewicz and P. N. Godbole, A penalty function approach to problems of plastic flow of metals with large surface deformation. *J. Strain Analysis* **10**, No. 3, 180–185, I. Mech. E. (1975).

5. O. C. Zienkiewicz and P. N. Godbole, Viscous incompressible flow with special reference to non-Newtonian (plastic) flow. *Finite Element Method in Flow Problems* (Edited by R. H. Gallagher, J. T. Oden, C. Taylor and O. C. Zienkiewicz), Chap. 2. Wiley, New York (1975).
6. P. N. Godbole, Creeping flow in rectangular ducts by the finite element method, *Int. J. Num. Meth. Engng* **9**, 727–731 (1975).
7. O. C. Zienkiewicz, Visco-plasticity, plasticity, creep and visco-plastic flow (problems of small, large and continuing deformation). *Int. Conf. Computational Methods in Non-linear Mechanics*, Texas (1974).
8. C. H. Lee and S. Kobagashi, New solutions to rigid-plastic deformation problems using a matrix method. *I. Eng. Ind. Trans. A.S.M.E.* **95**, 865 (1973).
9. G. C. Cornfield and R. H. Johnson, Theoretical prediction of plastic flow in hot rolling including the effect of various temperature distributions. *J. Iron and Steel Institute* **211**, 567–573 (1973).
10. J. W. H. Price and J. M. Alexander, The finite element analysis of two high temperature metal deformation processes. *2nd Int. Symp. Finite Element Methods in Flow Problems*, Santa Margherita Ligure (Italy), 715–728 (June 1976).
11. R. S. Dunham, On application of the finite element method to limit analysis. *Computational Methods in Non-linear Mechanics, International Conference on Computational Methods in Non-linear Mechanics*, Texas (1974).
12. P. Perzyna, Fundamental problems in visco-plasticity. *Recent Advances in Applied Mechanics*, Chap. 9, pp. 243–377. Academic Press, New York (1966).
13. O. C. Zienkiewicz and I. C. Cormeau, Visco-plasticity, plasticity and creep in elastic solids: A unified numerical solution approach. *Int. J. Num. Meth. Engng* **8**, 821–845 (1974).
14. O. C. Zienkiewicz, *The Finite Element Method in Engineering Science*. McGraw-Hill, London/New York (1971) (see also *The Finite Element Method*, 3rd Edn, 1977).
15. L. R. Hermann, Elasticity equations for incompressible and nearly incompressible materials by variational theorem. *AIAA J.* **3**, 1896–1900 (1965).
16. O. C. Zienkiewicz, Constrained variational principles and penalty function methods in the finite element analysis. *Lecture Notes in Mathematics*, No. 363, pp. 207–214. Springer-Verlag, Berlin (1974).
17. T. J. R. Hughes and H. Allik, Finite element of compressible and incompressible continua. *Proc. Sym., Civ. Eng.*, Vanderbilt Univ., (A.S.C.E. Publication), pp. 27–62 (1969).
18. P. Hood and C. Taylor, Navier-Stokes equations using mixed interpolation. *Finite Element Method in Flow Problems* (Edited by J. T. Oden, O. C. Zienkiewicz, R. H. Gallagher and C. Taylor), p. 121–132. UAH Press, Huntsville (1974).
19. D. J. Naylor, Stresses in nearly incompressible materials for finite elements with application to the calculation of excess pore pressure. *Int. J. Num. Meth. Engng* **8**, 443–460 (1974).
20. O. C. Zienkiewicz and E. Hinton, Reduced integration, function smoothing and non-conformity in finite element analysis (with special reference to thick plates). *J. Franklin Inst.* **302**, 443–461 (1976).
21. T. J. R. Hughes, R. L. Taylor and J. F. Levy, A finite element method for incompressible viscous flows. *Second Int. Symp. Finite Elements in Flow Problems*, Santa Margherita Ligure (Italy), **1** (June 1976).
22. R. E. Nickell, R. I. Tanner and B. Caswell, The solution of viscous incompressible jet and free surface flows using finite element methods. *J. Fluid Mech.* **65**, part 1, 189–206 (1974).
23. M. A. Matovich and J. R. A. Pearson, Spinning a molten threadline. Steady state isothermal viscous flows. *Industrial and Engineering Chemistry Fundamentals* **8**, No. 3, 512–520 (1969).
24. R. I. Tanner, R. E. Nickell and R. W. Bilger, Finite element methods for the solution of some incompressible non-Newtonian fluid mechanics problems with free surfaces. *Comp. Meth. Appl. Mech. and Engng* **6**, 155–174 (1975).
25. J. M. Alexander, On the theory of rolling. *Proc. R. Soc. Lond. A* **326**, 535–563 (1972).
26. H. Ford and J. M. Alexander, Simplified hot rolling calculations. *J. Inst. Met.* **4**, 92, 397 (1963).
27. A. K. Cruden, The cold extrusion of steel tubes and cans. The effect of punch Pilot size, Billet length/diameter ratio and base/wall thickness ratio. *N.E.L. Report No.* 472 (1970).
28. I. Christie, D. F. Griffiths, A. R. Mitchell and O. C. Zienkiewicz, Finite element methods for second order differential equations with significant first derivatives. *Int. J. Num. Methods in Engng* **10**, 1389–1396 (1976).
29. J. C. Heinrich, P. S. Huyakorn, A. R. Mitchell and O. C. Zienkiewicz, An "UPWIND" finite element scheme for two dimensional convective transport equation. *Int. J. Num. Meth. Engng* **11**, 131–143 (1977).
30. J. C. Heinrich and O. C. Zienkiewicz, Quadratic finite element schemes for two dimensional convective-transport problems. *Int. J. Num. Methods in Engng.*
31. J. F. W. Bishop, An approximate method for determining the temperatures reached in steady motion problems of plane plastics strain. *Quart. Jnl. Mech. and Applied Math.* **IX**, part 2, 236–246 (1956).